

Quantum Integrability for Three-Point Functions

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Quantum corrections to three-point functions of scalar single trace operators in planar $\mathcal{N} = 4$ Super-Yang-Mills theory are studied using integrability. At one loop, we find new algebraic structures that not only govern all two loop corrections to the mixing of the operators but also automatically incorporate all one loop diagrams correcting the tree level Wick contractions. Speculations about possible extensions of our construction to all loop orders are given. We also match our results with the strong coupling predictions in the classical (Frolov-Tseytlin) limit.

I. INTRODUCTION

In this paper we consider three-point correlation functions (3pt CF) of single trace gauge invariant operators of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. We consider mostly the first quantum correction (one loop) to the leading result (tree level) of [1] and speculate about some all loop features at the very end. The motivation for this study is twofold. On the one hand, the knowledge of the spectrum [2] together with the 3pt CF will suffice to determine any correlation function in this $3 + 1$ dimensional quantum field theory in a completely non-perturbative fashion. Such a highly ambitious goal is believed to be attainable due to the *Integrability*, or exact solvability, of planar $\mathcal{N} = 4$ SYM [2]. Another motivation is to better understand Holography and the emergence of a dual string description of a quantum gauge theory. How do smooth string worldsheets come about? Do they have a natural Integrable description in $\mathcal{N} = 4$ SYM? 3pt CF might be a great playground for addressing some of these questions. In particular, as we will reinforce in this letter, the answer to the last question seems to be *yes*; three-point functions can be studied most efficiently using integrability.

II. TWO LOOP EIGENSTATES

To compute the correlation functions at one loop we need to solve the two loop mixing problem. This is the subject of the current section. As in [1], we consider operators made out of two complex scalars (which are identified with states with \uparrow and \downarrow spins) that diagonalize the dilatation operator [3]

$$\hat{H} = (2g^2 - 8g^4) \sum_{i=1}^L \mathbb{H}_{i,i+1} + 2g^4 \sum_{i=1}^L \mathbb{H}_{i,i+2} + \mathcal{O}(g^6). \quad (1)$$

Here $\mathbb{H}_{a,b} \equiv \mathbb{I} - \mathbb{P}_{a,b}$ with \mathbb{P} being the permutation operator and sites $L + 1$ and 1 are identified. The fundamental excitations are magnons (spins \downarrow) moving in a ferromagnetic vacuum (where all spins are \uparrow). Their

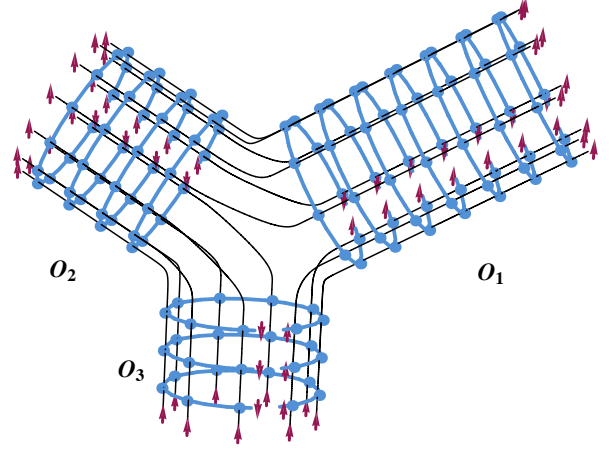


FIG. 1. Tree level correlation function of three single trace operators. Each operator \mathcal{O}_i is obtained by acting on a vacuum with a set of N_i creation operators (blue thick lines). This generates a state with L_i spins (thin black lines), N_i of which are flipped. These states are then glued together. We end up with a vertex model partition function with the topology of a thrice punctured sphere; it strongly resembles a discrete string path integral. We have $N_1 = N_2 + N_3$ so all spins \downarrow from \mathcal{O}_2 and \mathcal{O}_3 are contracted with \mathcal{O}_1 . Since there are N_3 thin lines connecting \mathcal{O}_1 and \mathcal{O}_3 all those lines are \downarrow spins; see [1] for details.

energy and momentum are parametrized as $E(u) = 2ig^2(1/x^+ - 1/x^-)$ and $p(u) = i \log(x^-/x^+)$ where the Zhukowsky variables $x^\pm = (u \pm i/2) - g^2/(u \pm i/2) + \mathcal{O}(g^4)$. The simplest state diagonalizing (1) is the single magnon

$$\sum_{n=1}^L \left(\frac{x^+}{x^-} \right)^n |\underbrace{\uparrow \dots \uparrow}_{n-1} \downarrow \uparrow \dots \uparrow\rangle. \quad (2)$$

At leading order in perturbation theory, there is an equivalent description of the states using the algebraic Bethe ansatz formalism (see [1] for a review). E.g., the single magnon state (2) simplifies to

$$\sum_n \left(\frac{u + i/2}{u - i/2} \right)^n \sigma_n^- |\uparrow \dots \uparrow\rangle \propto \hat{B}(u) |\uparrow \dots \uparrow\rangle \quad (3)$$

where the creation operators are given by

$$\hat{B}(u) = \uparrow \begin{array}{c} u \\ | \\ \text{---} \bullet \text{---} \end{array} \begin{array}{c} | \\ 0 \\ | \end{array} \begin{array}{c} | \\ 0 \\ | \end{array} \cdots \begin{array}{c} | \\ 0 \\ | \end{array} \begin{array}{c} | \\ 0 \\ | \end{array} \downarrow \quad (4)$$

with the R-matrix given by

$$\begin{array}{c} c \\ | \\ \text{---} \bullet \text{---} \\ | \\ \theta \\ | \\ d \end{array} \begin{array}{c} u \\ | \\ \text{---} \bullet \text{---} \end{array} \begin{array}{c} | \\ b \\ | \end{array} = \delta_{ab} \delta_{cd} + \frac{i}{u - \theta - \frac{i}{2}} \delta_{ad} \delta_{cb}$$

The algebraic treatment reveals its elegance when we consider states with N interacting magnons. This multi-particle state is simply given by

$$\hat{B}(u_1) \dots \hat{B}(u_N) |\uparrow \dots \uparrow\rangle. \quad (5)$$

Each of the legs in figure 1 corresponds to one such state. The energy of these states is given by $\sum E(u_i) = 2g^2 \Gamma_{\mathbf{u}}$,

$$\Gamma_{\mathbf{u}} = \sum_{i=1}^N \frac{1}{u_i^2 + \frac{1}{4}} + \mathcal{O}(g^2).$$

At tree level we should contract the states as in fig. 1 [1].

At the next loop order we need to improve (5) to obtain the two loop spin chain eigenstates. There are no explicit expressions for these eigenstates in the literature. We will now describe how to construct them using a modification of the algebraic Bethe ansatz. From (3) we see that we want to modify the propagation of the magnon along the chain to get the correct dispersion relation. The simplest way to achieve this preserving integrability is to introduce impurities θ_j at each site converting (4) into

$$\hat{B}(u) = \uparrow \begin{array}{c} u \\ | \\ \text{---} \bullet \text{---} \end{array} \begin{array}{c} | \\ \theta_1 \\ | \end{array} \begin{array}{c} | \\ \theta_2 \\ | \end{array} \cdots \begin{array}{c} | \\ \theta_{L-1} \\ | \end{array} \begin{array}{c} | \\ \theta_L \\ | \end{array} \downarrow \quad (6)$$

With these modified creation operators, the single magnon state $\hat{B}(u) |\uparrow \dots \uparrow\rangle$ takes the form

$$\sum_{n=1}^L \left[\prod_{k=1}^{n-1} \frac{u - \theta_k + \frac{i}{2}}{u - \theta_k - \frac{i}{2}} \right] \frac{i}{u - \theta_n - \frac{i}{2}} |\uparrow \dots \uparrow \downarrow \dots \uparrow\rangle_{n-1}. \quad (7)$$

The idea is to use the impurities θ_k to realize the required correction to the dispersion which arises at two loops. To achieve this we introduce the differential operator

$$\left(\frac{f}{\theta} \right) \equiv f + \frac{g^2}{2} \sum_{i=1}^L (\partial_{\theta_i} - \partial_{\theta_{i+1}})^2 f \Big|_{\theta_j \rightarrow 0} + \mathcal{O}(g^4) \quad (8)$$

which we call the Θ -derivative. Here ∂_{L+1} is identified with ∂_1 . It is easy to verify that applying the Θ -derivative to (7) we reproduce the good state (2) modulo a simple mismatch at the boundaries for the $n = 1, L$ terms in (2). What is way more remarkable is that, not only that mismatch can be fixed, but in fact,

$$(1 - g^2 \Gamma_{\mathbf{u}} \mathbb{H}_{L,1}) \left(\hat{B}(u_1) \dots \hat{B}(u_N) |\uparrow \dots \uparrow\rangle \right)_{\theta} \quad (9)$$

yields perfect N -magnon eigenstates of the two loop $\mathcal{N} = 4$ SYM dilatation operator [4]!

III. 3PT FUNCTIONS WITH IMPURITIES

The contractions between operators \mathcal{O}_3 and the other two operators are trivial, see caption of figure 1. The ones between \mathcal{O}_3 and \mathcal{O}_2 are simply contractions of $L_3 - N_3$ \uparrow spins while the contractions between \mathcal{O}_3 and \mathcal{O}_1 involve N_3 \downarrow spins. That is, the effect of the operator \mathcal{O}_3 is to remove a piece of ferromagnetic vacuum of length $L_3 - N_3$ from \mathcal{O}_2 and replace it with a sequence of magnons of length N_3 . In formulas, $|2\rangle \equiv \hat{B}(v_1) \dots \hat{B}(v_{N_2}) |\uparrow\rangle^{\otimes L_2} \rightarrow \hat{\mathcal{O}}_3 |2\rangle$ where [5]

$$\hat{\mathcal{O}}_3 = \left(|\downarrow\rangle^{\otimes N_3} \right) \left({}^{\otimes L_3 - N_3} \langle \uparrow | \right). \quad (10)$$

The operator $\hat{\mathcal{O}}_3 |2\rangle$, of length L_1 , should be contracted with \mathcal{O}_1 given by $|1\rangle \equiv \hat{B}(u_1) \dots \hat{B}(u_{N_1}) |\uparrow\rangle^{\otimes L_1}$. For simplicity, we will consider the case where the third operator \mathcal{O}_3 is a chiral primary. Then, the (absolute value of the properly normalized) tree level 3pt function with impurities is simply [1, 6]

$$|C_{123}^{\text{tree with imp.}}| = \frac{\sqrt{L_1 L_2 L_3}}{\sqrt{\binom{L_3}{N_3}}} \frac{|\langle 1 | \hat{\mathcal{O}}_3 | 2 \rangle|}{\sqrt{\langle 1 | 1 \rangle \langle 2 | 2 \rangle}}. \quad (11)$$

Let us specify which impurities we use in (6) when constructing $|1\rangle$ and $|2\rangle$. Each thin line in figure 1 has its own impurity. The impurities associated to the contractions between operator \mathcal{O}_n and \mathcal{O}_m are denoted by $\{\theta_j^{nm}\}$. We define $\{\theta_j^1\} = \{\theta_j^{12}\} \cup \{\theta_j^{13}\}$ etc. Explicit expressions for the scalar products in (11) are presented in the appendix. The tree level result C_{123}^{tree} in $\mathcal{N} = 4$ SYM is given by (11) if we send to zero all impurities. The impurities will be important when extending this expression to one loop.

IV. ONE LOOP 3PT FUNCTIONS

When computing 3pt CF at one loop, two effects need to be taken into account: (a) we need to correct the one loop operators into the two loop Bethe eigenstates and (b) add insertions of Hamiltonians at the splitting points [8]. The first effect leads to (11) where we replace the one loop states by the two loop eigenstates constructed via (9) and indicated by boldface,

$$|C_{123}^{\text{one loop (a)}}| = \frac{\sqrt{L_1 L_2 L_3}}{\sqrt{\binom{L_3}{N_3}}} \frac{|\langle \mathbf{1} | \hat{\mathcal{O}}_3 | \mathbf{2} \rangle|}{\sqrt{\langle \mathbf{1} | \mathbf{1} \rangle \langle \mathbf{2} | \mathbf{2} \rangle}}. \quad (12)$$

To compute this quantity we start with a tree level scalar product with impurities such as $\langle 1 | 1 \rangle$. Then we act with the Θ -derivative (8) on it. When this differential operator acts on $|1\rangle$ we get $|\mathbf{1}\rangle$ up to a simple boundary term (9). Same is true for $\langle 1 |$. Then we also have the crossed terms when one of the derivatives in (8) acts on $|1\rangle$ and another one acts on $\langle 1 |$. These can be dealt with using

$$i (\partial_{\theta_j} - \partial_{\theta_{j+1}}) \hat{B}(u) \Big|_{\theta \rightarrow 0} = \left[P_{j,j+1} + \delta_{j,L} \sum_{i=1}^L \mathbb{H}_{i,i+1}, \hat{B}(u) \right]$$

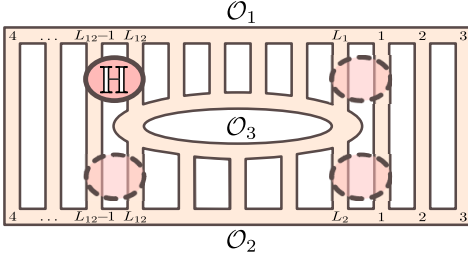


FIG. 2. To take into account the loop diagrams correcting the Wick contractions of the operators one must insert Hamiltonian densities at the junctions of the operators [8].

At the end of the day, we find [4]

$$\langle 1|1 \rangle = [1 - g^2 (\Gamma_u^2 + 2\Gamma_u)] \left(\langle 1|1 \rangle \right)_{\theta^1}$$

and an analogous expression for $\langle 2|2 \rangle$. Similarly, for the numerator, we find

$$\begin{aligned} |\langle 1|\hat{O}_3|2 \rangle| = & \left| \left[1 - \frac{g^2}{2} (\Gamma_u^2 + 2\Gamma_u + \Gamma_v^2 + 2\Gamma_v) \right] \left(\langle 1|\hat{O}_3|2 \rangle \right)_{\theta^1} \right. \\ & + g^2 \langle 1|\mathbb{H}_{L_{12}-1, L_{12}} \hat{O}_3|2 \rangle + g^2 \langle 1|\hat{O}_3 \mathbb{H}_{L_{12}-1, L_{12}}|2 \rangle \\ & \left. + g^2 \langle 1|\mathbb{H}_{L_1, 1} \hat{O}_3|2 \rangle + g^2 \langle 1|\hat{O}_3 \mathbb{H}_{L_2, 1}|2 \rangle \right| \quad (13) \end{aligned}$$

where $L_{12} = L_1 - N_3$. For the last two lines we should set the impurities to zero. Two remarkable things happen when we put everything together. First, all the Γ_u and Γ_v cancel out when we construct the ratio (12). Second, the last two lines in (13) are nothing but Hamiltonian insertions at the splitting points (see figure 2). They cancel *precisely* with the Hamiltonian insertions which come from adding up all Feynman diagrams correcting the tree level Wick contractions [8]! As such, when the dust settles, we end up with our main result

$$|C_{123}^{\text{one loop}}| = \frac{\sqrt{L_1 L_2 L_3}}{\sqrt{\binom{L_3}{N_3}}} \frac{\left| \left(\langle 1|\hat{O}_3|2 \rangle \right)_{\theta^1} \right|}{\left(\sqrt{\langle 1|1 \rangle} \right)_{\theta^1} \left(\sqrt{\langle 2|2 \rangle} \right)_{\theta^2}} \quad (14)$$

for the structure constants up to one loop [7]. The striking simplicity of this result signals a deeper structure which the Θ -derivative starts to unveil. The derivatives in (14) can be explicitly computed with ease [4].

V. COMPARISON WITH STRING THEORY

The strong coupling regime of $\mathcal{N} = 4$ SYM theory is described by classical strings. Our results are, strictly speaking, valid at weak coupling. Yet, we shall demonstrate that in a particular limit they coincide precisely with the string theory results.

The limit where one can in principle expect a match is the Frolov-Tseytlin limit [9]. This is the limit of large operators $L_i \sim N_i \rightarrow \infty$ but with $g/L_i \ll 1$. We will use the results of [10] where $O_1 \simeq O_2^{\dagger}$ correspond to two

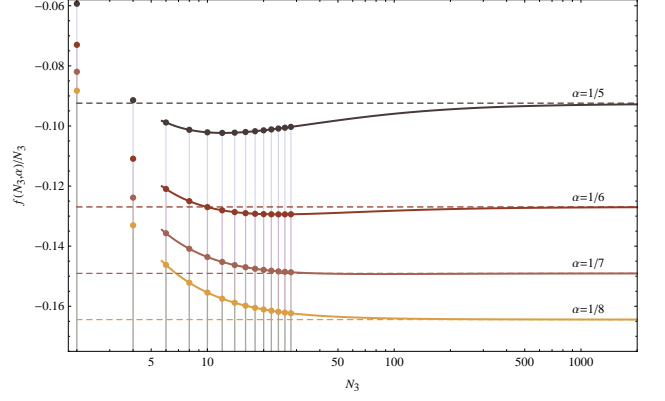


FIG. 3. $f(N_3, \alpha)/N_3$ for several α 's as a function of N_3 . The solid lines are fits in (large) N_3 . The dashed lines are the strong coupling string theory results (16). The fits asymptote to the dashed lines within the numerical accuracy. To build this figure we considered in total about 1000 combinations of three states with up to 56 magnons and lengths as large as 450. This computation would be absolutely inconceivable without our main result (14).

similar classical strings while O_3 is a small BPS string. The closest we can get to the Frolov-Tseytlin limit for all operators is then

$$1 \ll N_3, L_3 \ll g \ll L_1, L_2, N_1, N_2. \quad (15)$$

This is the limit we consider. As in [11], we will use the $SU(2)$ folded string solution which is simple enough to work with and has a rich structure at the same time. We also take $L_3 = 2N_3$ for the small operator O_3 . The result is then a function of 3 parameters only: $\alpha \equiv N_1/L_1$, L_1 and N_3 . The tree level weak coupling result matches the leading order expansion in g/L_1 of the string theory result, denoted as C_{123}^{tree} [11] (see also [12]). For the next order, we find

$$\frac{C_{123}^{\text{string}}}{C_{123}^{\text{tree}}} \simeq 1 + \frac{g^2 N_3}{L_1^2} \left[\frac{32\alpha(1-2q)E^2(q)}{(\alpha-1)(\alpha^2-2\alpha q+q)} + \mathcal{O}\left(\frac{1}{N_3}\right) \right] \quad (16)$$

where $q(\alpha)$ is related to α via $\alpha = 1 - E(q)/K(q)$. A remarkable feature of this *strong coupling* result is that it resembles a *weak coupling* expansion in g^2 .

To compare with this result we found the corresponding solution of the two loop Bethe ansatz equations for several values of L_1, N_1 and N_3 with very high numerical precision (see [11] for details on the g^0 Bethe roots). Then we plug the Bethe roots into (14) and extrapolate the result to infinite length by increasing N_1 and L_1 with fixed ratio α . We find that the one loop correction (normalized by the tree level result) decays indeed as $f(N_3, \alpha)/L_1^2$ as L_1 goes to infinity. The values of $f(N_3, \alpha)$ for various N_3, α are shown in fig. 3. We observe that $f(N_3, \alpha)$ increases linearly with N_3 . To compare with (16) we found the leading linear term in N_3 with a fit. Note that a priori it is not obvious at all that the weak

coupling result (14) scales as N_3/L_1^2 . The fact that it does is already highly remarkable and encouraging. Of course, even more striking is the fact that the coefficient matches precisely the string result (16)!, see fig. 3.

Curiously, at tree level the weak and strong coupling results match for any finite N_3 [11]. The numerical analysis at one loop indicates that there is no agreement for finite N_3 ; only the leading term in large N_3 matches (16).

VI. CONCLUSIONS AND MUSING

There is a longstanding idea that the complexity of the long-range integrable structure of the AdS/CFT system might come from integrating out some hidden degrees of freedom [13, 14]. The impurities θ_j and the Θ -derivative realize this idea at weak coupling. Particularly inspiring is the fact that the Θ -derivative not only corrects the states but it also automatically incorporates all one loop Feynman diagrams involved in gluing together the three operators.

As we saw, the Θ -derivative naturally leads to the Zhukowsky variables. For example the norm $(\langle 1|1 \rangle)_{\theta^1}$ takes the form (A1) where in ϕ_k we replace [4]

$$\prod_{a=1}^{L_1} \frac{u_k - \theta_a^{(1)} + i/2}{u_k - \theta_a^{(1)} - i/2} \rightarrow \left(\frac{x_k^+}{x_k^-} \right)^{L_1}. \quad (17)$$

This leads to the natural guess that, to all loops, we should simply deform the dispersion and S-matrix in ϕ_k as in the spectrum problem. The same comments hold for the main part of the numerator of (14), the matrix G_{nm} written in the Appendix. Hence, with some insight from the spectrum problem, with the help of the Θ -derivative method, and with the inspiration of the Inozemtsev approach [14], we believe that a conjecture for the all loops structure constants might be within reach for asymptotically large operators. A first step could be to understand in detail the single magnon case which was so fruitful at

one loop. For example, if in (8) we have

$$\mathcal{O}(g^4) = \frac{g^4}{8} \sum_{|i-j| \neq 1} (\partial_{\theta_i} - \partial_{\theta_{i+1}})^2 (\partial_{\theta_j} - \partial_{\theta_{j+1}})^2 f + \mathcal{O}(g^6)$$

then the action of the Θ -derivative on the single magnon state (7) yields (2) up to three-loop order modulo simple boundary terms. We believe that the same holds for multi-particle states. Then, a natural conjecture is that (14) holds up to two loops. This being investigated [16]. At higher loops, one could try to incorporate the dressing phase using the boost operator of [15].

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Appendix A: Formulae for Scalar Products

$$\langle 1|1 \rangle = \prod_{m \neq k} \frac{u_k - u_m + i}{u_k - u_m} \det_{j,k \leq N_1} \frac{\partial \phi_k}{\partial u_j} \quad (A1)$$

with $e^{i\phi_k} = \prod_{a=1}^{L_1} \frac{u_k - \theta_a^{(1)} + i/2}{u_k - \theta_a^{(1)} - i/2} \prod_{m \neq k}^{N_1} \frac{u_k - u_m - i}{u_k - u_m + i}$ and similar

for $\langle 2|2 \rangle$. Finally [6] $\langle 1|\hat{\mathcal{O}}_3|2 \rangle = \mathcal{F} \det \left([G_{nm}] \oplus [F_{nm}] \right)$

where $F_{nm} = \frac{1}{(u_n - \theta_m)^2 + \frac{1}{4}}$,

$$G_{nm} = \prod_{a=1}^L \frac{v_m - \theta_a^{(1)} + i/2}{v_m - \theta_a^{(1)} - i/2} \frac{\prod_{k \neq n}^{N_1} (u_k - v_m + i)}{u_n - v_m} - \frac{\prod_{k \neq n}^{N_1} (u_k - v_m - i)}{u_n - v_m},$$

$$\mathcal{F} = \frac{\prod_m^{N_3} \prod_n^{N_1} (u_n - \theta_m^{(1)} + i/2) / \prod_m^{N_3} \prod_n^{N_2} (v_n - \theta_m^{(1)} + i/2)}{\prod_{n < m}^{N_1} (u_m - u_n) \prod_{n < m}^{N_2} (v_n - v_m) \prod_{n < m}^{N_3} (\theta_n^{(1)} - \theta_m^{(1)})}.$$

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